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Some Properties of Strongly Principally Self-Injective Modules

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Abstract

The idea of generalizing quasi injective by employing a new term is introduced in this paper. The introduction of principally self-injective modules, which are principally self-injective modules. A number of characteristics and characterizations of such modules have been established. In addition, the idea of strongly mainly self-pure sub-modules was added, which is similar to strongly primarily self-injective sub-modules. Some characteristics of injective, quasi-injective, principally self-injective, principally injective, absolutely self-pure, absolutely pure, and finitely *R*-injective modules being lengthened to strongly principally self-injective modules. So, in the present work, some properties are added to the concept in a manner similar to the absolutely self-neatness. The fundamental features of these concepts and their interrelationships are linked to the conceptions of some rings. (Von Neumann) regular, left SF-ring, and left pp-ring rings are described via such concept. For instance, the homomorphic picture of every principally injective module be strongly principally self-injective if *R* being left pp-ring, and another example for a commutative ring R of every strongly principally self-injective module be flat if *R* being (Von Neumann) regular. Also, a ring *R* be (Von Neumann) regular if and only if each *R*-module being strongly principally self-injective module.

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1. Introduction

A module M is injective which is stated by M. F. Hamid [1], that is equal to Baer condition [2], which states that a module A is injective if each R-homomorphism $f: I \rightarrow A$ can be stretched to an R-homomorphism $h: R \rightarrow A$.for each left ideal I of R. The concept of injective modules has been extensively researched, and several generalizations have been offered. For example, R. E. Johnson and E. T. Wong developed quasi injective modules as a generalization of injective modules [3], while Nicholson introduced principal injective as a generalization of injective modules in 1995 [4]. A module M is said to be ***quasi injective*** in [5] and [6]. From the other side, L. Fuchs in 1969 [7] verified that a module M is quasi injective if and only if for each left ideal I of R, and each R-homomorphism $\alpha: I \rightarrow M$, with ker $\alpha \in \Omega(M)$, where the ker α means kernel of the map α, ($\Omega(M)$ = set of the whole left ideals I of R, then $I \supseteq \text{ann}(m)$ for certain $m \in M$, where the ann(m) means annihilator (m)), there is an R-homomorphism $\gamma: R \rightarrow M$ lengthening α. That's similar to the Baer state for the injective modules.

The minimum (quasi) injective module containing M as a sub-module is named the (quasi) injective ***envelope*** of M by [2, 5].

In what follows, R denotes an associative ring with identity, and all modules are left unitary R-modules. There is another concept that is also of interest in [1], which is of ***absolutely self-pure*** modules. Consequently, every absolutely pure module is finitely ***R-injective*** [8], and the absolute self-purity idea generalizes the quasi-injectivity, finite R-injectivity, and absolute purity [5]. The purpose of this study is to present a concept that sits between absolute self-purity and principal self injectivity and generalize this concept in a way that is similar to absolutely self-neatness [9]. A module M is called ***strongly principally self-injective*** (denoted sps-injective) module if it is strongly principally self-pure in each module containing it as a sub-module.

The main problem that was raised is. We do not know whether direct sum two principally self-injective modules are direct sum or not. In order to find a solution to this problem, we presented a sub-concept paving the way to present our main concept. A sub-module M of an R-module N is called ***strongly principally self-pure*** sub-module of N (Denoted $M \leq^{sps-p} N$) when the subsequent being satisfied: For each principal left ideal I of R as well as each R-homeomorphism $\alpha: I \to M$, with $\ker \alpha \in \overline{\Omega}(M)$, there's also an R-homomorphism $\gamma : R \to N$ causing the following commutative diagram 1.

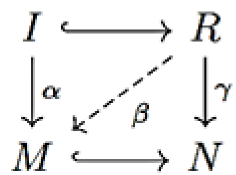

Diagram 1: Strongly principally self-pure.

After that, there's an R-homomorphism $\beta : R \to M$ causing the higher triangle to be commutative. Some examples of this concept are provided. Also, some properties of strongly principally self-pure sub-modules are added, among them, it is proved that a strongly principally self-pure sub-module of a module is transitive.

The main goal of this study is to introduce strongly principally self-injective modules, which are located between absolutely self-pure modules and principally self-injective modules. Through this concept, we proved that the direct sum of two strongly principally self-injective modules are strongly principally self-injective modules. As well, some rings are characterized by using strongly principally self-injective modules and strongly principally pure sub-modules. Employing such idea, (Von Neumann) regular as well as left pp-ring is characterized. Where these rings were formerly distinguished by multiple concepts, for example, rings were distinguished as (Von Neumann) regular rings by (co)pure see [10-12]. Genetic algorithms are studied in cryptography see [13], which is one of the important algebraic applications. Interestingly, some rings were distinguished as pp-ring rings by p-injective module see [14-17]. Where we proved every module is a sps-injective module if and only if a ring R be (Von Neumann) regular and R be left pp-ring if and only if the homomorphic image of any absolutely pure module is sps-injective module.

2. Methodology of Research

In this section, some of the necessary concepts, which are related to the present work, are reviewed. For example, injective, quasi injective and other concepts are related to the current work:

2.1. Definition [2] A module M is called ***injective*** if for every R-monomorphism $\gamma: A \to B$ of modules and every R-homomorphism $\alpha: A \to M$, there is an R-homomorphism $\beta: B \to M$, then $\alpha = \gamma \circ \beta$. That's similar to the Baer state [2] and the module A being injective if and only if, for each left ideal I of R, each R-homomorphism $f: I \to A$ can be lengthened to an R-homomorphism $h: R \to A$.

2.2. Definition [5, 6] A module M is called ***quasi injective*** when for each sub-module $N \subseteq M$ and each R-homomorphism $\alpha: N \to M$, can be lengthened to a M endomorphism.

2.3. Definition [2] and [5] The minimum (quasi) injective module containing M as a sub-module is called the (quasi) injective M ***envelope***.

2.4. Definition [1, 17] A module M is said to be ***principally injective*** (denoted p-injective) when every R-homomorphism from an R principal left ideal to M can be extended to an R-homomorphism from $R \to M$. It's obvious that each injective module being principally injective as well as quasi injective, but the opposite isn't correct.

2.5. Definition [18] A sub-module M of a module N is said to be ***pure sub-module*** when for the following commutative diagram 2, with a finitely generated sub-module I of a free module.

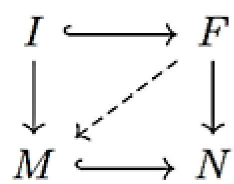

Diagram 2: Pure sub-module.

There is a R-homomorphism $F \to M$ causing the higher triangle be commutative.

2.6. Definition [19, 20] A module M is called ***absolutely pure*** when it's pure in every module that contains it as a sub-module.

2.7. Definition [21] A module M is said to be ***absolutely self-pure*** when for each finitely created left ideal I of R as well as each R-homomorphism $\alpha: I \to M$ with ker $\alpha \in \overline{\Omega}(M)$, there is an extension $\gamma: R \to M$ of α.

2.8. Definition [8] A module M is called ***finitely R-injective*** when each R-homomorphism from a finitely created left ideal of $R \to M$ can be lengthened to an R-homomorphism $R \to M$. Consequently, every absolutely pure module is finitely R-injective. The absolute self-purity idea generalizes the finite R-injectivity, quasi infectiveness, and absolute purity.

2.9. Definition [9] A module M is called ***neat*** in N, precisely for the following commutative diagram 3.

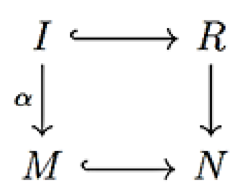

Diagram 3: Neat module.

Where, I be the greatest left ideal of R, and there's an R-homomorphism $R \to M$ that extends α.

2.10. Definition [9] A module M is called ***absolutely self-neat*** when for each R-homomorphism $\alpha: I \to M$, where I is the greatest left ideal of R and ker $\alpha \in \overline{\Omega}(M)$, there's an R-homomorphism $\gamma: R \to M$ that extends α.

2.11. Definition A sub-module M of a module N is called ***strongly principally self-pure*** of N (Denoted $M \leq_{sps-p} N$) when the subsequent being correct: For each principal left ideal I of R as well as each R-homeomorphism $\alpha : I \to M$, with ker $\alpha \in \overline{\Omega}(M)$, when there's also an R-homomorphism $\gamma: R \to N$ making the following commutative diagram 4.

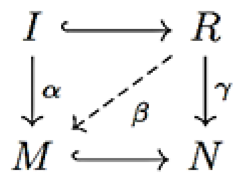

Diagram 4: Strongly principally self-pure module.

After that, there's an R-homomorphism $\beta : R \rightarrow M$ causing the higher triangle be commutative.

2.12. Definition A module M is called ***strongly principally self-injective*** (denoted sps-injective) when being strongly principally self-pure sub-module in each R-module containing it.

3. Main Results

In this section, the basic properties of strongly principally self-pure sub-module and strongly principally self-injective module by some new results are proved, and some examples of these two concepts are given.

The following result describes a strongly principally self-pure sub-module.

3.1. Proposition Let $A \subseteq B \subseteq C$ are modules.

(1) If $A \leq^{sps-p} B$ and $B \leq^{sps-p} C$, then $A \leq^{sps-p} C$.

(2) If $A \leq^{sps-p} C$, then $A \leq^{sps-p} B$.

Proof. 1. Consider the following commutative diagram 5.

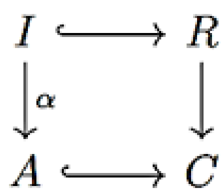

Diagram 5: A is sps-pure sub-module in C.

Where, I represent the principal left ideal of R, and α represents the R-homomorphism, with ker $\alpha \in \overline{\Omega}(A)$, and Y: $R \rightarrow C$ is an R-homomorphism making the diagram commutative. One can regard α as R-homomorphism $I \rightarrow B$ for getting the following commutative diagram 6.

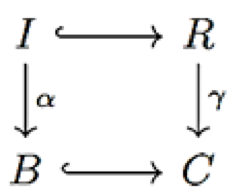

Diagram 6: B is sps-pure sub-module in C.

As $B \leq^{sps-p} C$, there's an R-homomorphism Y: $R \rightarrow B$ of α, therefore the following commutative diagram 7:

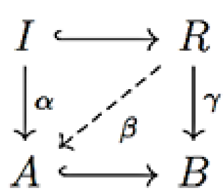

Diagram 7: A is sps-pure sub-module in B.

And since $A \leq^{sps-p} B$, there's an R-homomorphism β: $R \to A$ that extends α.

2. Take into consideration the following commutative diagram 8.

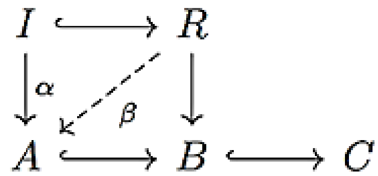

Diagram 8: *A* is sps-pure sub-module.

Since $A \leq^{sps-p} C$, there's an R-homomorphism β: $R \to A$ that extends α.

3.2. Examples

1) Any pure sub-module of a module is strongly principally self-pure but not the other way around, and because of any strongly principally self-pure in it is injective envelope, but not pure.

2) Any self-pure sub-module of an R-module being strongly principally self-pure.

The following result proves that the strongly principally self-pure sub-module is a form of injective module.

3.3. Proposition An R-module N is strongly principally self-injective when for every principal left ideal I of R, as well as every R-homomorphism α: $I \to N$, with ker $\alpha \in \overline{\Omega}(N)$, there's an R-homomorphism $R \to M$ that extends α.

Proof. For any R-homomorphism α: $I \to N$ as above, there's an R-homomorphism γ: $R \to Q(N)$ making the following commutative diagram 9:

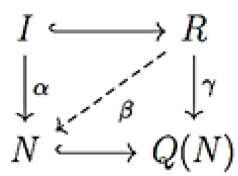

Diagram 9: *N* is sps-pure sub-module in Q(*N*).

But, N is strongly principally self-injective if and only if it's strongly principally self-pure in $Q(N)$, if and only if, there's an R-homomorphism β: $R \to N$ that extends α. ■

3.4. Examples

1) Any (quasi) injective module is clearly sps-injective module.

2) Every p-injective module is clearly sps-injective module.

3.5. Proposition Each sps-pure sub-module of an sps-injective module being once more sps-injective.

Proof. Take into consideration the following commutative diagram 10.

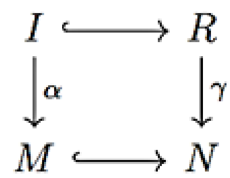

Diagram 10: *N* is sps-injective module.

When *N* be sps-injective, where ker $\alpha \in \overline{\Omega}$ (*M*), then there is an *R*-homomorphism *Ύ*: $R \to N$ causing the drawing be commutative. If $M \leq^{sps-p} N$, there's an *R*-homomorphism $R \to M$ causing the higher triangle be commutative and extending α.

3.6. Examples

1) Every absolutely (self) pure module is sps-injective.

2) The $\mathbb{Z}$-modules ($\mathbb{Z}_n$) are sps-injective module but not a p-injective.

The subsequent corollary considers the preceding proposition.

3.7. Corollary Each direct summand of a sps-injective module is once more sps-injective.

Property of strongly principally self-infectiveness is preserved when taking the finite direct sums of a module property of strongly principally self-infectiveness. This can be proved by the following Theorem:

3.8. Theorem A module *A* being strongly principally self-injective module if and only if $A \oplus A$ be strongly principally self-injective module.

Proof. If $A \oplus A$ is strongly principally self-injective module, then *A* is strongly principally self-injective module, since it is the result of a direct summand $A \oplus A$. Conversely, if *A* is strongly principally self-injective, then for any *R*-homomorphism $\alpha : I \to A \oplus A$, where *I* is a principal left ideal of *R*, and ker $\alpha \in \overline{\Omega}(A \oplus A)$, we have ker $\alpha \supseteq \cap_{i=1}^{I} \text{ann}(x_i, y_i) = \cap_{i=1}^{I} \text{ann}(x_i) \cap \cap_{i=1}^{I} \text{ann}(y_i)$, for some $x_i, y_i \in A$, $i = 1, \ldots, I$. This means that ker $\alpha \in \overline{\Omega}(A)$. Having, $\alpha = \alpha_1 \oplus \alpha_2$, where α_1 and α_2 are obtained by following α as a result of the natural projections of $A \oplus A$ onto $A \oplus 0$ and $0 \oplus A$, respectively, we see that every of ker α_1, and ker α_2, contains ker α. So, they must be in $\overline{\Omega}(A)$. by strongly principally self-infectiveness of *A*, there are *R*-homomorphism β_1 and $\beta_1 : R \to A$ extending α_1 and α_2, respectively. Now, $\beta_1 \oplus \beta_2$ is the desired extension of α. ■

Recall that *M* of a module *N* is called ***principally self-pure*** sub-module of *N* [23] (denoted $A \leq^{ps-p} B$) when the subsequent be correct: For each principal left ideal *I* of *R* as well as each *R*-homomorphism $\alpha: I \to M$, with ker $\alpha \supseteq \text{ann}(m)$ for certain $m \in M$, there's also an *R*-homomorphism *Ύ*: $R \to N$ making the following commutative diagram 11.

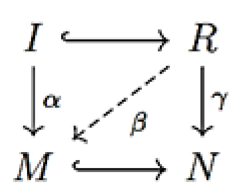

Diagram 11: Principally self-pure sub-module.

After that, there's an *R*-homomorphism $\beta : R \to M$, causing the higher triangle be commutative.

Also, a module *M* is called ***principally self-injective*** [23] (denoted ps-injective) when for every principal left ideal *I* of *R* as well as every *R*-homomorphism $\alpha: I \to M$, with ker $\alpha \supseteq \text{ann}(m)$ for certain $m \in M$, there's an *R*-homomorphism $R \to M$ that extends α.

3.10. Remarks

1. Every sps-pure sub-module is a ps-pure sub-module, but the opposite is not necessarily true.

2. Every sps-injective module is a ps-injective module, but the opposite is not necessarily true.

3. When a module M be strongly principally self-pure in certain quasi injective module by Proposition 2.3, it has to be sps-pure in its quasi-injective envelope $Q(M)$. When M is a sub-module of module N, the $Q(M) \leq^{sps-p} Q(M)$, and one has to pose, by Proposition 2.3, that $M \leq^{sps-p} N$. From the above theorem proof, a module being sps-injective, if and only if it's strongly principally self-pure in certain quasi injective module if and only if it's strongly principally self-pure in its quasi-injective envelope.

The following figure 1, shows the relationship among all these concepts, (quasi) injective modules with ps-injective, absolutely (self) pure, principally injective, and sps-injective.

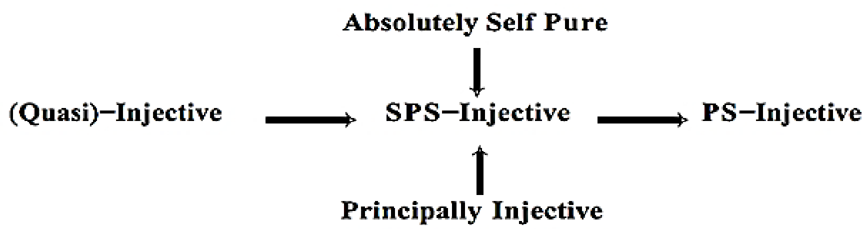

Figure 1: Strongly principally self-injective modules.

The above figure 1, states that the concept of sps-injective is generalization of injective, quasi-injective, absolutely self-pure, and principally injective modules. It also shows that the concept of sps-injective is generalized by the ps-injective module.

4. Characterization of Some Rings

In this section, some new results of the characterization of some rings by sps-injective are shown. Where, every module being sps-injective if R is (Von Neumann) regular, the homomorphic image of each is absolutely pure module being sps-injective if and only if R is left pp-ring, and for a commutative ring R, each sps-injective module be flat equivalent to a ring R that being (Von Neumann) regular.

By recalling that a ring R is called ***left principally projective ring*** (denoted left pp-ring) [5] when every of its principal left-ideals being projective.

One can characterize the left pp-rings in the following theorem after the Lemma.

4.1. Lemma [21] A left ideal I in a ring R being projective if and only if for each epimorphism $M \to M'$ from an injective module M, and each R-homomorphism $R \to M$, with $\ker \alpha \in \overline{\Omega}(M)$ can be lifted to an R-homomorphism $I \to M$.

4.2. Theorem A ring R is a left pp-ring if and only if the homomorphic image of any injective module is sps-injective.

Proof. ($\Rightarrow$) Let M be injective module and let $\beta : M \to N$ is each epimorphism, one will verify that N being sps-injective. Each principal left ideal I of R being projective, the inclusion map is $\iota : I \to R$ as well as $\Upsilon : I \to$ N. Via supposing, a ring R being left pp-ring. That involves that there's an R-homomorphism α: $R \to M$ that can be raised to an R-homomorphism $\delta : R \to N$, consequently, there's δ that extends γ. Therefore, N is sps-injective.

For ($\Leftarrow$), One will verify each principal left ideal of R be projective. Take into consideration the following commutative diagram 12.

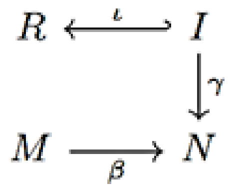

Diagram 12: Homomorphic image of the N injective module.

Where, I represents a principal left ideal of R, and the inclusion map is $\iota : I \rightarrow R$ s well as $\Upsilon : I \rightarrow N$ of any R-homomorphism with ker $\alpha \in \overline{\Omega}$ (N) and $\beta : M \rightarrow N$, is an epimorphism from an injective module M. By assumption, N is sps-injective. That involves that there's an R-homomorphism $\delta : R \rightarrow N$ that extends γ. But, R being projective, consequently, there's an R-homomorphism $\alpha : R \rightarrow M$ that lifts δ.

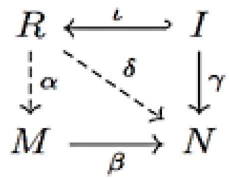

Diagram 13: A ring R is a left pp-ring.

So, one has $\beta\alpha\iota = \delta\iota = \gamma$, and I is projective via Lemma 3.1. ■

A ring R is called (***Von Neumann) regular*** (see [24, 26]) if each principal left ideal of R being a straight summand. Over a (Von Neumann) regular ring, every module is sps-injective. Also, it's correct in the opposite direction.

The following theorem proved that every module is a sps-injective over (Von Neumann) regular rings.

4.3. Theorem A ring R is (Von Neumann) regular if and only if each module is a sps-injective.

Proof. (⇒) Every module is absolutely self-pure modules [5], therefore it's sps-injective.

(⇐) Consider the following commutative diagram 13.

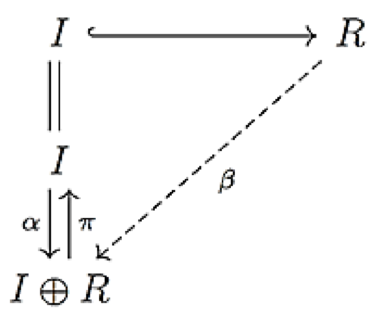

Diagram 13: A module is sps-injective.

Let I represents a principal left ideal of R, also the identity map of I to extend by R-homomorphism $\alpha : I \rightarrow I \oplus R$ is described via $x \mapsto (x, 0)$. One will prove this map is well defined, let $x \in I$, $x \mapsto (x, 0)$, ann(x) = 0, ann(x) ≤ ann(x, 0), therefore α is well defined. By assumption, $I \oplus R$ is sps-injective, hence there is an R-homomorphism β: $R \rightarrow I \oplus R$ extending β, with ker $\alpha \in \overline{\Omega}(I \oplus R)$. There is a β by the projection π: $I \oplus R \rightarrow I$ to extend the identity map $I \rightarrow I$, and as proof I is the direct summand in R, this was proven by Corollary 2.8.

Recall that a ring R is called ***left SF-ring*** [2] when each simple left module being flat. One recognizes that a left module A is called ***flat*** if and only if A^{+} is injective [27, Proposition 3.54, p.136].

In the following theorem, the equivalence (1) ⟺ (2) is proved if R is commutative.

4.4. Theorem For a commutative ring R, the following are equivalent:

(1) R represents the (Von Neumann) regular ring.

(2) Each sps-injective module being flat.

Proof. (1) ⇒ (2) is trivial. For (2) ⇐ (1), one will verify that R being (Von Neumann) regular. Each easy module is a quasi-injective [2]. Therefore, each easy module is sps-injective, also each sps-injective module being flat via the supposition. Consequently, R represents a SF-ring. Thus, R represents the (Von Neumann) regular [6, Theorem 3.16].

4.5. Example In [23], there's an example of 1-injective ring R (= p-injective) that isn't 2-injective; this means there's a left ideal I of R, created via 2-elements as well as an R-homomorphism $I \rightarrow R$, without extending to $R \rightarrow R$. It is clear that this is an instance of a sps-injective module but it is not absolutely self-pure.

5. Applications About Modules

Modules theory has been used in telecommunications applications, provides space-time coding, and the design of signal constellations for multi-antenna radio transmission. Therefore, the group representation theory is based on module theory, it is necessary to mention everything that is based on it. For example, the usage of modules across an algebraic number field aids the signal constellation design in telecommunications engineering and theoretical physics. The representation theory of groups and modules are inextricably linked. They are also a key concept in commutative and homological algebra, and they're utilized a lot in algebraic geometry and algebraic topology.

6. Conclusions

The idea of strongly primarily self-injective modules was presented in this paper, which lies in between absolute self-purity and principal self-injectivity. R is von Neumann regular only if every module being primarily self-injective, and R is pp-ring only when the homomorphic image of every completely pure module is principally self-injective. The idea of a sub-module that is strongly principally self-pure sub-module was proposed. It was established that the relationships between principally self-pure sub-modules are transitive as the relationships between strongly principally self-pure sub-modules. This is similar to what is found in a pure sub-module.

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Conflict of Interest

The researchers state that they possess no conflict of interest.

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