Journal of Applied Sciences and Nanotechnology

The connection between Graphs' Chromatic and Ehrhart Polynomials

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A B S T R A C T

Graph Theory is a discipline of mathematics with numerous outstanding issues and applications in various sectors of mathematics and science. The chromatic polynomial is a type of polynomial that has valuable and attractive qualities. Ehrhart's polynomials and chromatic analysis are two essential techniques for graph analysis. They both provide insight into the graph's structure but in different ways. The relationship between chromatic and Ehrhart polynomials is an area of active research that has implications for graph theory, combinatorial, and other fields. By understanding the relationship between these two polynomials, one can better understand the structure of graphs and how they interact. This can help us to solve complex problems in our lives more efficiently and effectively. This work gives the relationship between these two essential polynomials and the proof of theorems, and an application related to these works, the model Physical Cell ID (PCID), was discussed.

1. Introduction

A pair $G = (V; E)$ comprising a set V and a multiset $E \subseteq \{\{i, j\}: i, j \in V\}$, is called a graph G. Elements that are vertices or nodes of G make up the set V. The edges of G are in the set E [1] discovered the chromatic polynomial in 1912 while attempting to prove the four-coloring theorem. Since the four-colour theory was first hypothesized in 1852, proving it would have been a significant event. Graph colouring G with n −coloring is a map $\emptyset: V \rightarrow \{1, 2, \ldots, n\}$ with dimension n, and the set of colours $\{1, 2, \ldots, n\}$. A proper n-colouring of the vectors $\emptyset: V \rightarrow \{1, 2, \ldots, n\}$ is mapped with the colouring $\emptyset(i) \neq \emptyset(j)$ whenever i and j are adjacent. The number of appropriate graph colourings is counted as a function of n colours by the chromatic polynomial $\chi(G; n)$[2]. Richard Stanley first proposed counting order-preserving maps and its connection to chromatic polynomials in 1970. In other words, given a finite set $\{v_1, v_2, \ldots, v_n\} \subset \mathbb{R}^d$, the smallest convex set containing those points is the polytope P. P is the convex hull of a finite number of points in $\mathbb{R}^d$. We refer to P^o is interior as P. For some $k \in \mathbb{Z}_{>0}$ and polytope P, kP is a dilated polytope if $kP = \{(kx_1, kx_2, \ldots, kx_d): (x_1, x_2, \ldots, x_d) \in P\}$. In Ehrhart's theory, the number of lattice points in integral dilates of P is investigated using lattice polytope in $\mathbb{R}^d$ with d dimensions. For $k \in \mathbb{Z}_{>0}$, the lattice point enumerator for the kth dilate is $E_P(k) = \#(kP \cap \mathbb{Z}^d)$. The polynomial $E_P(k)$ has degree k, with rational coefficients known as the Ehrhart polynomial [3], A lot of applications can be found on [4,5,6]. The order polynomial is studied in two areas of mathematics: algebraic graph theory and algebraic combinatorics. The order polynomial counts the number of order-preserving maps from a poset to an n-dimensional chain. Richard P. Stanley created these order-preserving maps in 1971 while working on his Ph.D. at

Harvard University under the direction of Gian-Carlo Rota and researching ordered structures and partitions. In [7] it presents two different geometric explanations for P. Although the order polytope is our primary concern, these two polytopes share some intriguing findings. By maintaining an acyclic orientation, a link may be established between the chromatic polynomials of graph G and the Ehrhart polynomials of the order polytope compatible with various poset in G. The relationship between chromatic polynomials and Ehrhart polynomials has been studied extensively in the field of mathematics, leading to a variety of applications. This relationship has been used to solve problems in graph theory, combinatorics, and geometry, among other areas. [8-10]. It has also provided insight into the properties of convex polytopes and their associated graphs. By understanding the underlying principles behind this relationship, researchers have been able to develop algorithms that can be used to solve various problems related to chromatic polynomials and Ehrhart polynomials. [11-13]. Some of the most important applications of this relationship and how they can be used to solve various real-world problems were discussed. This article defines partially ordered groups and the connection between non-periodic modes and graphs. The reciprocity theorem, and order polynomials $\Omega(n)$ are also discussed, in this work, theorems including the connection between order polytope and chromatic polynomials are given. Additionally, the series' numerator polynomial P_Π is decomposing as,

$$\sum \Omega^o(n)z^n = \frac{P_\Pi(z)}{(1-z)^{d+1}} \tag{1}$$

were P_Π is a polynomial with a nonnegative integer coefficient.

2. Preliminaries
This section presents some essential ideas and theorems pertinent to our work.

Definition 1[7]: A set Π with a binary relation $R \subseteq \Pi \times \Pi$ that satisfies three requirements is referred to as a partially ordered set or poset.

- Reflexivity, if $a \in \Pi$ then $a \leqslant a$.
- Ant symmetry, if $a \leqslant b$ and $b \leqslant a$, then $a = b$
- and Transitivity, if $a \leqslant b$ and $b \leqslant c$ then $a \leqslant c$.

Definition 2 [1]: Each pair of vertices in a complete graph is connected by an edge.

Definition 3 [7]: An orientation of G is the direction designation by the symbols i→j or j→i to each edge ij. If there are no coherently directed cycles in an orientation of G, it is acyclic. Fig. 1a, and b shows that K_3 with an acyclic orientation and a noncyclic orientation, respectively.
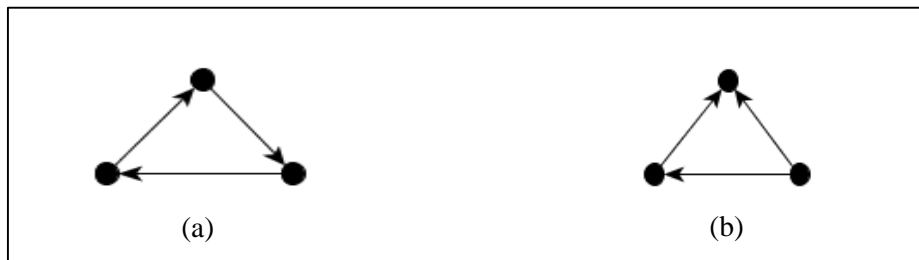

(a)   (b)

Figure 1: (a) an acyclic orientation of K_3, (b) a noncyclic orientation.

Definition 4[7]: Let Π denote a finite poset and n denote an Γ_n length chain. A chain is an ordered set, specifically the set $\{1,2,\ldots,n\}$ with the natural order. A map $\emptyset : \Pi \to \Gamma_n$ is order preserving if $x \leq y$ implies that $\emptyset(x) \leq \emptyset(y)$. Let $\Omega_\Pi(n)$ be the number of such order-preserving maps, and the order polynomial $\Omega(n) = \Omega(p,n)$ is the function that counts their number.

Definition 5 [7]: an order polynomial that counts the number of strictly order-preserving maps: $\Pi \to \Gamma_n$, meaning x<y implies $\emptyset(x) < \emptyset(y)$. Let $\Omega_\Pi^o(n)$ be the number of such strict-order polynomials, then $\Omega^o(n) = \Omega^o(p, n)$.

Definition 6 [14]: A symmetric polynomial is a polynomial where if it switches any pair of variables, it remains the same. For example, $x^2 + y^2 + z2\, x^2 + y^2 + z^2 x^2 + y^2 + z^2$ is a symmetric polynomial, since switching any pair, say x and y, the resulting polynomial $y^2 + x^2 + z2\, y^2 + x^2 + z^2 y^2 + x^2 + z^2$ is the same as the initial polynomial

Definition 7 [15]: The set of points $(x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$ with $0 \le x_i \le 1$ and if $t_i \le p\, t_j$ then $x_i \le x_j$ is the order polytope $\mathcal{O}(\Pi)$, of a poset Π with elements $\{t_1, t_2, \ldots, t_n\}$, each point $(x_1, x_2, \ldots, x_n)$ of $\mathcal{O}(\Pi)$ is identified with the function $f: \Pi \longrightarrow \mathbb{R}$, $f(t_i) = x_i$.

Definition 8 [16]: A series of rational numbers known as the Bernoulli numbers B n frequently appears in analysis. The Euler-Maclaurin formula, Faulhaber's formula for the sum of the m-th powers of the first n positive integers, Taylor series expansions of the tangent and hyperbolic tangent functions, and expressions for specific Riemann zeta function values can all be used to define the Bernoulli numbers B_n. The table below contains a list of the first 20 Bernoulli numbers.

Table 1: Bernoulli numbers.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
B_n	0	$\frac{1}{2}$	$\frac{1}{6}$	0	$\frac{-1}{30}$	0	$\frac{1}{42}$	0	$\frac{1}{30}$	0	$\frac{5}{66}$	0	$\frac{-691}{2730}$	0	$\frac{7}{6}$	0	$\frac{3617}{510}$	0	$\frac{3867}{798}$	0	$\frac{-174611}{330}$

Theorem 1[7]: Order-preserving maps and strictly order-preserving maps are connected that is,

$$(\mathbf{n}) = (-\mathbf{1})^{|\Pi|}\mathbf{\Omega}(-\mathbf{n}) \tag{2}$$

This restores the negative binomial identity in the case of a chain. The chromatic polynomial and the Ehrhart polynomial are exceptional cases of Stanley's general Reciprocity Theorem yield similar results.

Theorem 2[17]: The formulas relate the polynomials $\Omega_\Pi(n)$ and $\Omega_\Pi^o(n)$ to the Ehrhart polynomial of the order polytope $\mathcal{O}_\Pi$ are,

$$\mathbf{E}_{\mathcal{O}_\Pi}(\mathbf{n}) = \mathbf{\Omega}_\Pi(\mathbf{n} + \mathbf{1}), \mathbf{E}_{\mathcal{O}_\Pi^o(n)} = \mathbf{\Omega}_\Pi^o(\mathbf{n} - \mathbf{1}) \tag{3}$$

Theorem 3[7]: The sum of the strict order polynomials for all acyclic orientations σ of a graph G is the chromatic polynomial $\chi_n(G)$.

$$\mathbf{\chi_n(G)} = \sum_{\sigma\ \textbf{acyclic}} \mathbf{\Omega}_\Pi^o(\mathbf{n}) \tag{4}$$

Theorem 4[18]: For any nonnegative integer k, let Π be the poset with one minimal element covered by k other elements. It has the Ehrhart polynomial of the order polytope $\mathcal{O}(\Pi)$,

$$\mathbf{E}(\mathcal{O}(\Pi), \mathbf{t}) = \mathbf{1} + \sum_{j=1}^{k} \frac{(B_{k-j+1} + (k-j+1))}{k-j+1} \binom{k}{j} \mathbf{t}^j + \frac{1}{k+1} \mathbf{t}^{k+1} \tag{5}$$

Were, B_n is the nth Bernoulli number.

Theorem 5[18]: For any positive integer j satisfying $1 \le j \le k-1$, the coefficient of t j of $E(\mathcal{O}(\Pi), t)$ is negative if and only if $k - j + 1 \ge 20$ and 4 divides $k - j + 1$. Hence, the order polytope $\mathcal{O}(\Pi)$ is Ehrhart positive if and only if $k < 20$.

3. Main Results

The relationship between chromatic and Ehrhart polynomials has been studied extensively in mathematics. This section will explore the proofs of theorems that link these two polynomials. We will look at how chromatic polynomials can be used to determine Ehrhart polynomials and vice versa. We will also discuss the use cases of these theorems in various contexts.

Theorem 6: Let G be a graph with n vertices, $\chi(G)$ define the Ehrhart polynomial as $E_{\mathcal{O}_\Pi}$, then the relationship between chromatic polynomial and Ehrhart polynomial for graphs is:

$$\chi_n(G) = \sum_{\sigma \text{ acyclic}} (-1)^{|\Pi|} E_{\mathcal{O}_\Pi}(-n-1) \tag{6}$$

Proof: By using theorem 1

$$\Omega^o(n) = (-1)^{|\Pi|}\Omega(-n)$$

$$E_{\mathcal{O}_\Pi}(n) = \Omega_\Pi(n+1) \rightarrow E_{\mathcal{O}_\Pi}(n-1) = \Omega_\Pi(n)$$

$$E_{\mathcal{O}_\Pi^o(n)} = \Omega_\Pi^o(n-1).$$

So, $E_{\mathcal{O}_\Pi}(n-1) = \Omega_\Pi(n)$

$$E_{\mathcal{O}_\Pi}(-n-1) = \Omega_\Pi(-n)$$

$$(-1)^{|\Pi|} E_{\mathcal{O}_\Pi}(-n-1) = (-1)^{|\Pi|}\Omega_\Pi(-n) = \Omega_\Pi^o(n).$$

Then, $(-1)^{|\Pi|} E_{\mathcal{O}_\Pi}(-n-1) = \Omega_\Pi^o(n) , \chi_n(G) = \sum_{\sigma \text{ acyclic}} \Omega_\Pi^o(n)$

This is the result.

Theorem 7: Let $\mathcal{O}(\Pi)$ be an order polytope of dimension d and let $\mathcal{P}_\Pi$ be the numerator polynomial of the series

$$\sum \Omega^o(n)z^n = \frac{\mathcal{P}_\Pi(z)}{(1-z)^{d+1}} \tag{7}$$

Then $\mathcal{P}_\Pi$ can be decomposed as $\mathcal{P}_\Pi(z) = a(z) + zb(z)$ where the polynomials a(z) and b(z) are symmetric with $a(z) = z^d a(\frac{1}{z})$ and $b(z) = z^{d-1}(\frac{1}{z})$, and so that b(z) and $-a(z)$ have nonnegative coefficients.

Proof: By using theorem (2)

$$E_{\mathcal{O}_\Pi}(n) = \Omega_\Pi(n+1) \tag{8}$$

$$\text{And theorem (1) } \Omega^o(n) = (-1)^{|\Pi|}\Omega(-n) \tag{9}$$

The following were obtained

$$E_{\mathcal{O}_\Pi^o(n)} = \Omega_\Pi^o(n-1) \tag{10}$$

$$\text{Put them in } \sum \Omega^o(n)z^n = \frac{\mathcal{P}_\Pi(z)}{(1-z)^{d+1}} \tag{11}$$

The results depend on the Bernoulli formula

$$E(\mathcal{O}(\Pi), t) = 1 + \sum_{j=1}^{k} \frac{(B_{k-j+1}+(k-j+1))}{k-j+1}\binom{k}{j}t^j + \frac{1}{k+1}t^{k+1} \qquad (12)$$

According to the relationship between Bernoulli and Eulerian numbers

$$E_n = \sum_{k=0}^{n}\binom{n}{k}(2^{k+1}-2^{2k+2})\frac{B_{k+1}}{k+1}) \qquad (13)$$

$$\text{So } h^*(z) = \sum_{k=0}^{d-1} E(d,k)z^k \qquad (14)$$

Were the coefficients of the polynomial given by Eulerian numbers that count the number of permutations of $\{1,\dots,n\}$ which have k descent., [19]. This means that it gives a symmetric polynomial that satisfying.

$$h^*(z) = z^{d-1}h^*\left(\frac{1}{z}\right) \qquad (15)$$

So the numerator $⟦\, \mathcal{P}_\Pi(z)$ is equal $z^d h^*\left(\frac{1}{z}\right) = z\, h^*(z)$, so by decomposition for $\mathcal{P}_\Pi(z)$, a(z)=0 and b(z)= $h^*(z)$, that satisfies the required conditions.

For example, to find the Ehrhart polynomial $E(\mathcal{O}(\Pi), t)$ of order polytope $\mathcal{O}(\Pi)$, of dim $= k+1$, for $k = 4$:

$$E(\mathcal{O}(\Pi), t) = 1 + \sum_{j=1}^{4} \frac{(B_{4-j+1}+(4-j+1))}{4-j+1}\binom{4}{1}t^j + \frac{1}{4+1}t^{4+1} \qquad (16)$$

$$= 1 + \frac{B_4+4}{4}\binom{4}{1}t + \frac{B_3+3}{3}\binom{4}{2}t^2 + \frac{B_2+2}{2}\binom{4}{3}t^3 + \frac{B_1+1}{1}\binom{4}{4}t^4 + \frac{1}{4+1}t^{4+1}$$

$$= 1 - \frac{1}{3}t + 6t^2 + \frac{13}{3}t^3 + \frac{3}{2}t^4 + \frac{1}{5}t^5$$

Moreover, by direct computation, we see the linear coefficient of E. $(\mathcal{O}(\Pi), t)$ equals $-\frac{1}{3}t$, which is negative.

4. Application for the Ehrhart and Chromatic Polynomials
The problem of graph colouring in large, intricate networks like social and informational ones, on the other hand, is a fundamental component of many applications in which the goal is to divide a set of entities into classes where two related entities are not in the same class while also minimizing the number of classes used, despite being useful in a variety of fields (e.g., engineering, scientific computing) [20, 21]. The application, therefore, is given to discuss our idea.

4.1 Numerical Computation Application
A femtocell is a small, low-power cellular base station typically designed for home or small business use. Small cell is a broader term that is more widespread in the industry, with femtocell as a subset. The model (PCID) allocation problem with random graph colouring problem. Bernoulli's random graph model is proposed to model deployment scenarios of femtocells, which is applied to allocate PCIDs efficiently, a graph coloring-based PCIDs allocation scheme. Model allocation of PCID as a random Graph colourization problem in femtocellular networks. To solve the problem of PCID allocation with minimal computational expense, the chromatic polynomial of random graphs or order polytopes should be used so that cells overlap femtocells that are coloured differently to avoid a collision. As a result, Figure 2-b depicts each femtocell with its own set of f nodes. An undirected edge connects neighbouring nodes. Second-order neighbours are connected to the same edges to prevent network confusion constraints. The colour will not map until a second-order node is reached, as illustrated in Fig. 2c. 3D shape depicts a scenario in which any algorithm can colourize the graph to avoid collisions and confusion. The number of colours required to colour all of the graph's vertices.
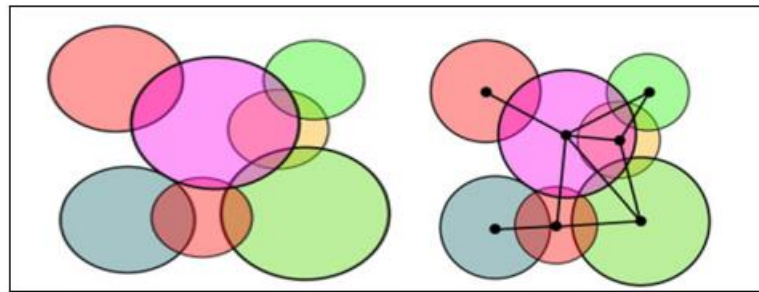
(a) Femtocellular LTE-A network topology

(b) Each femtocellular is represented by a node. Neighboring nodes are connected via edges.
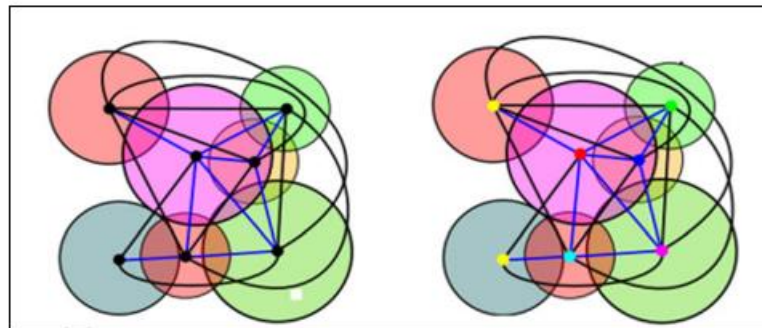
(c) To avoid confusion 2nd-degree neighbors are connected via the edges algorithm.

(d) Now PCID assignment can be done by a graph coloring

Figure 2: PCID allocation in femtocellular networks is transformed into a graph colouring algorithm.

5. Conclusions

Graph theory has a wide range of areas that depend on the properties of graphs, which are in combinatorial optimization, graph colouring fundamental issues with many applications, such as timetabling and scheduling, frequency assignment, register allocation, and more recently, the analysis of networks of human subjects. The relationship between chromatic and Ehrhart polynomials is meaningful in combinatorial mathematics. This work provided proof of theorems that relate the two polynomials, allowing us to understand better how they are related. We describe how each polynomial's coefficients are described, their respective properties, and how these can be used to prove various results. By understanding these relationships, we can more effectively use chromatic and Ehrhart polynomials to solve problems in combinatorics. More works using this relationship were found in [22].

Conflict of Interest
The authors declare that they have no conflict of interest.

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